

BAREM DE EVALUARE ȘI DE NOTARE
Varianta 71

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♦ Pentru orice soluție corectă, chiar dacă este diferită de cea din barem, se acordă punctajul corespunzător.

♦ Nu se acordă fracțiuni de punct, dar se pot acorda punctaje intermediare pentru rezolvări parțiale, în limitele punctajului indicat în barem.

♦ Se acordă 10 puncte din oficiu. Nota finală se calculează prin împărțirea punctajului obținut la 10.

SUBIECTUL I (30 de puncte)

1.	$m = 5^{1+x} + 5^{1-x} = 5 \cdot 5^x + 5 \cdot 5^{-x} = 5(5^x + 5^{-x}) = 5\left(5^x + \frac{1}{5^x}\right)$ $p = 25^x + 25^{-x} = 5^{2x} + 5^{-2x} = (5^x)^2 + (5^{-x})^2 = (5^x)^2 + \left(\frac{1}{5^x}\right)^2 = \left(5^x + \frac{1}{5^x}\right)^2 - 2 \cdot 5^x \cdot \frac{1}{5^x} = \left(5^x + \frac{1}{5^x}\right)^2 - 2$ $\text{Dar } m+p=2n \Rightarrow \begin{cases} 5\left(5^x + \frac{1}{5^x}\right) + \left(5^x + \frac{1}{5^x}\right)^2 - 2 = a \\ \text{Not. } 5^x + \frac{1}{5^x} = y \end{cases} \Rightarrow y^2 + 5y - 2 = a \Leftrightarrow y^2 + 5y - 2 - a = 0$ <p>Ec. are rădăcini reale $\Leftrightarrow \Delta \geq 0$ (1); Cum $y > 0$ (2)</p> $\stackrel{(1)+(2)}{\Rightarrow} \begin{cases} \Delta \geq 0 \\ P > 0 \end{cases} \Leftrightarrow \begin{cases} 4a + 33 \geq 0 \\ -a - 2 > 0 \end{cases} \Leftrightarrow \begin{cases} a \geq -\frac{33}{4} \\ a < -2 \end{cases} \Rightarrow a \in \left[-\frac{33}{4}; -2\right)$	1p 1p 1p 2p
2.	$f(x) > 0 \Leftrightarrow \frac{mx^2 + 2(m+1)x + m - 2}{x^2 + 1} > 0 \Leftrightarrow mx^2 + 2(m+1)x + m - 2 > 0, \forall x \in \mathbb{R}$ $\Rightarrow \begin{cases} m > 0 \\ \Delta < 0 \end{cases} \Leftrightarrow \begin{cases} m > 0 \\ 4m + 1 < 0 \end{cases} \Leftrightarrow \begin{cases} m \in (0; +\infty) \\ m \in \left(-\infty; -\frac{1}{4}\right) \end{cases} \Rightarrow m \in (0; +\infty) \cap \left(-\infty; -\frac{1}{4}\right) = \emptyset$	1p 2p 2p
3.	$M = \{x / x \in \mathbb{Z}, x \leq 7\} = \{-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$ <p>Pt. $x \geq 0 \Rightarrow x^2 - 5x + 4 = 0 \Rightarrow x_1 = 4; x_2 = 1$</p> <p>Pt. $x < 0 \Rightarrow x^2 + 5x + 4 = 0 \Rightarrow x_1 = -4; x_2 = -1$</p> <p>Evenimentul $A = \{-4, -1, 1, 4\}$</p> <p>$\Rightarrow$ Evenimentul contrar lui A este: $\bar{A} = \{-7, -6, -5, -3, -2, 0, 2, 3, 5, 6, 7\}$</p>	1p 1p 1p 1p 1p
4.	$(17x^5 - 18y)^{2012} = C_{2012}^0 (17x^5)^{2012} (-18y)^0 + C_{2012}^1 (17x^5)^{2011} (-18y)^1 + \dots$ $\dots + C_{2012}^{2012} (17x^5)^0 (-18y)^{2012}$ <p>Suma coeficienților este:</p> $C_{2012}^0 17^{2012} (-18)^0 + C_{2012}^1 17^{2011} (-18)^1 + C_{2012}^2 17^{2010} (-18)^2 + \dots + C_{2012}^{2012} 17^0 (-18)^{2012} =$ $= (17 + (-18))^{2012} = (-1)^{2012} = 1$ <p>cf. form. binom. Newton</p>	2p 3p
5.	$\vec{a} = 3\vec{m} - 2\vec{n} + \vec{p} = 3(2\vec{i} - 3\vec{j}) - 2(-\vec{i} + 2\vec{j}) + (4\vec{j}) = 8\vec{i} - 9\vec{j}$ $\vec{b} = -2\vec{m} + \vec{n} - \vec{p} = -2(2\vec{i} - 3\vec{j}) + (-\vec{i} + 2\vec{j}) - 4\vec{j} = -5\vec{i} + 4\vec{j}$ $\vec{a} + \vec{b} = (8\vec{i} - 9\vec{j}) + (-5\vec{i} + 4\vec{j}) = 3\vec{i} - 5\vec{j} \Rightarrow \vec{a} + \vec{b} = \sqrt{9 + 25} = \sqrt{34}$	2p 2p 1p

6.	Folosim relația $\arcsin x = \operatorname{arctg} \frac{x}{\sqrt{1-x^2}}, x \in (0;1)$	2p
	$\arcsin \frac{4}{5} = \operatorname{arctg} \frac{\frac{4}{5}}{\sqrt{1-\frac{16}{25}}} = \operatorname{arctg} \frac{4}{3} \Rightarrow \operatorname{tg} \left(\arcsin \frac{4}{5} + \operatorname{arctg} \frac{7}{24} \right) =$	
	$= \operatorname{tg} \left(\operatorname{arctg} \frac{4}{3} + \operatorname{arctg} \frac{7}{24} \right) = \operatorname{tg} (a+b) \text{ unde am notăm } \begin{cases} \operatorname{arctg} \frac{4}{3} = a \\ \operatorname{arctg} \frac{7}{24} = b \end{cases} \Rightarrow \begin{cases} \operatorname{tga} = \frac{4}{3} \\ \operatorname{tgb} = \frac{7}{24} \end{cases}$	2p
	$\Rightarrow \operatorname{tg} (a+b) = \frac{\operatorname{tga} + \operatorname{tgb}}{1 - \operatorname{tga} \cdot \operatorname{tgb}} = \dots = \frac{117}{44}$	1p

SUBIECTUL al II-lea (30 de puncte)

1. a)	Elementele inversabile din $\mathbb{Z}_{12}$ sunt: $\hat{1}, \hat{5}, \hat{7}, \hat{11}$ $\Rightarrow \hat{1} + \hat{5} + \hat{7} + \hat{11} = \hat{0}$	3p 2p
b)	$\det A = \begin{vmatrix} \hat{1} & \hat{0} & \hat{5} \\ \hat{x} & \hat{1} & \hat{2} \\ \hat{4} & \hat{3} & \hat{1} \end{vmatrix} = \hat{3}x + \hat{11}$ $\Rightarrow \det A = \hat{3}x + \hat{11} \neq \hat{0} \Rightarrow \hat{3}x \neq \hat{1} \Rightarrow x \in \left\{ \hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}, \hat{6}, \hat{7}, \hat{8}, \hat{9}, \hat{10}, \hat{11} \right\} = \mathbb{Z}_{12}$ $\Rightarrow A \text{ este inversabilă } \forall x \in \mathbb{Z}_{12}.$	2p 2p
c)	Pentru $x = \hat{0} \Rightarrow A^{-1} = \begin{pmatrix} \hat{5} & \hat{9} & \hat{5} \\ \hat{4} & \hat{7} & \hat{2} \\ \hat{8} & \hat{3} & \hat{11} \end{pmatrix} \in M(\mathbb{Z}_{12})$ $\text{Ecuația } YA=B \Leftrightarrow Y=BA^{-1} = \begin{pmatrix} \hat{1} & \hat{0} & \hat{2} \\ \hat{2} & \hat{1} & \hat{0} \\ \hat{0} & \hat{2} & \hat{1} \end{pmatrix} \begin{pmatrix} \hat{5} & \hat{9} & \hat{5} \\ \hat{4} & \hat{7} & \hat{2} \\ \hat{8} & \hat{3} & \hat{11} \end{pmatrix} = \begin{pmatrix} \hat{5} & \hat{3} & \hat{3} \\ \hat{2} & \hat{1} & \hat{0} \\ \hat{4} & \hat{5} & \hat{3} \end{pmatrix}$	3p 2p
2. a)	$\left. \begin{aligned} f(1) &= a_0 + a_1 + a_2 + a_3 + \dots + a_{2n-1} + a_{2n} = 3^n \\ f(-1) &= a_0 - a_1 + a_2 - a_3 + \dots - a_{2n-1} + a_{2n} = 1 \end{aligned} \right\} \Rightarrow a_1 + a_3 + \dots + a_{2n-1} = \frac{f(1) - f(-1)}{2} =$ $= \frac{3^n - 1}{2}$	1p 3p 1p
b)	Cf. teor. împărțirii cu rest a polinoamelor $\Rightarrow$ $f = (X+2)^2 \cdot q + ax + b, a, b \in \mathbb{R} \Rightarrow f(-2) = -2a + b = 3^n (1)$	3p

	$f' = 2(X+2)q + (X+2)^2 q' + a \Rightarrow f'(-2) = a(2)$ <p>Dar $f' = n(X^2 + X + 1)^{n-1} (2X + 1) \Rightarrow f'(-2) = n \cdot 3^{n-1} \cdot (-3) = -n \cdot 3^n (3)$</p> <p>$\stackrel{\text{din (2) si (3)}}{\Rightarrow} a = -n \cdot 3^n$ de unde $b = 3^n (1 - 2n)$</p>	2p
c)	<p>$f(x) = f(-x) \Leftrightarrow (x^2 + x + 1)^n = (x^2 - x + 1)^n$; cum $x^2 - x + 1 \neq 0, \forall x \in \mathbb{R} \Rightarrow$</p> $\left(\frac{x^2 + x + 1}{x^2 - x + 1} \right)^n = 1 = \cos 2k\pi + i \sin 2k\pi \Rightarrow \frac{x^2 + x + 1}{x^2 - x + 1} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n};$ <p>$k \in \{0, 1, \dots, n-1\}$</p> <p>Pentru $k=0 \Rightarrow x_0=0$</p> <p>Pentru $k = \overline{1, n-1}$ avem</p> $\frac{x^2 + 1}{x} = \frac{1 + \cos \alpha + i \sin \alpha}{-1 + \cos \alpha + i \sin \alpha} \Leftrightarrow \frac{x^2 + 1}{x} = \frac{2 \cos \frac{\alpha}{2} \left(\cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2} \right)}{2i^2 \sin^2 \frac{\alpha}{2} + 2i \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = -i \operatorname{ctg} \frac{\alpha}{2}$ $\text{Deci } x^2 + ix \operatorname{ctg} \frac{\alpha}{2} + 1 = 0 \Rightarrow x_{1,2} = \frac{-i \left(\operatorname{ctg} \frac{k\pi}{n} \pm \sqrt{4 + \operatorname{ctg}^2 \frac{k\pi}{n}} \right)}{2}; k = \overline{1, n-1}$	<p>2p</p> <p>1p</p>

SUBIECTUL al III-lea (30 de puncte)

1. a)	$f'(x) = \frac{x+2}{x+1}$ $\lim_{\substack{x \rightarrow -1 \\ x > -1}} f(x) = -\infty; \lim_{x \rightarrow +\infty} f(x) = +\infty$ <p>Semnul lui $f'(x)$:</p> <table> <tr> <th>x</th><th>$-\infty$</th><th>-1</th><th>$+\infty$</th></tr> <tr> <td>$x+2$</td><td></td><td></td><td>+</td></tr> <tr> <td>$x+1$</td><td></td><td></td><td>+</td></tr> <tr> <td>$f'(x)$</td><td></td><td></td><td>+</td></tr> <tr> <td>$f(x)$</td><td></td><td></td><td>$-\infty \nearrow +\infty$</td></tr> </table> <p>$\Rightarrow f$ este strict crescătoare pe domeniul său de definiție.</p>	x	$-\infty$	-1	$+\infty$	$x+2$			+	$x+1$			+	$f'(x)$			+	$f(x)$			$-\infty \nearrow +\infty$	1p 1p 2p 1p
x	$-\infty$	-1	$+\infty$																			
$x+2$			+																			
$x+1$			+																			
$f'(x)$			+																			
$f(x)$			$-\infty \nearrow +\infty$																			

b)	<p>Se arată că pe intervalul $(-1;0)$ avem $f(x) < 0 \Rightarrow$</p> $A(\alpha) = -\int_{\alpha}^0 f(x) dx = -\int_{\alpha}^0 (x + \ln(1+x)) dx = -\int_{\alpha}^0 x dx - \int_{\alpha}^0 \ln(1+x) dx =$ $= -\frac{x^2}{2} \Big _{\alpha}^0 - \int_{\alpha}^0 x' \cdot \ln(1+x) dx = 0 + \frac{\alpha^2}{2} - \left[x \cdot \ln(1+x) \Big _{\alpha}^0 - \int_{\alpha}^0 \frac{x}{1+x} dx \right] =$ $= \frac{\alpha^2}{2} - \left[0 - \alpha \cdot \ln(1+\alpha) - \int_{\alpha}^0 \left(1 - \frac{1}{1+x} \right) dx \right] = \frac{\alpha^2}{2} + \alpha \cdot \ln(1+\alpha) + x \Big _{\alpha}^0 - \ln(1+x) \Big _{\alpha}^0 =$ $= \frac{\alpha^2}{2} + \alpha \cdot \ln(1+\alpha) - \alpha + \ln(1+\alpha) = \frac{\alpha^2}{2} - \alpha + (\alpha+1) \ln(\alpha+1),$	1p 3p 1p
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c)	$\lim_{\alpha \rightarrow -1} A(\alpha) = \lim_{\alpha \rightarrow -1} \left(\frac{\alpha^2}{2} - \alpha + (\alpha+1) \ln(\alpha+1) \right) = \lim_{\alpha \rightarrow -1} \left(\frac{\alpha^2}{2} - \alpha \right) + \underbrace{\lim_{\alpha \rightarrow -1} (\alpha+1) \ln(\alpha+1)}_{not.L} =$ $= \frac{3}{2} + L = \frac{3}{2} + 0 = \frac{3}{2}$ $\text{Unde } L = \lim_{\alpha \rightarrow -1} (\alpha+1) \ln(\alpha+1) = \lim_{\alpha \rightarrow -1} \frac{\ln(\alpha+1)}{\frac{1}{\alpha+1}} = \lim_{\alpha \rightarrow -1} \frac{1}{-\frac{1}{(\alpha+1)^2}} = \lim_{\alpha \rightarrow -1} -(1+\alpha) = 0$	2p 3p
2. a)	$I_0 = \int_1^e x dx = \frac{x^2}{2} \Big _1^e = \frac{e^2 - 1}{2}$ $I_1 = \int_1^e x \ln x dx = \int_1^e \left(\frac{x^2}{2} \right)' \ln x dx = \left(\frac{x^2}{2} \cdot \ln x \right) \Big _1^e - \int_1^e \frac{1}{x} \cdot \frac{x^2}{2} dx = \frac{e^2}{2} - \frac{1}{2} \int_1^e x dx =$ $= \frac{e^2}{2} - \frac{1}{2} \cdot \frac{e^2 - 1}{2} = \frac{e^2 + 1}{4}$	2p 3p
b)	<p>Folosim metoda integrării prin părți $\Rightarrow$</p> $I_n = \int_1^e x (\ln x)^n dx = \int_1^e \left(\frac{x^2}{2} \right)' (\ln x)^n dx = \left(\frac{x^2}{2} (\ln x)^n \right) \Big _1^e - n \int_1^e (\ln x)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^2}{2} dx =$ $= \frac{e^2}{2} - \frac{n}{2} \int_1^e x (\ln x)^{n-1} dx = \frac{e^2}{2} - \frac{n}{2} \cdot I_{n-1}$ <p>Am găsit $I_n = \frac{e^2}{2} - \frac{n}{2} \cdot I_{n-1} \Big \cdot 2 \Rightarrow 2I_n + nI_{n-1} = e^2$</p>	3p 2p
c)	<p>Șirul $(I_n)_{n \geq 0}$ este descrescător $\Rightarrow$</p> $\begin{cases} I_{n+1} - I_n \leq 0 \\ I_{n+1} - I_n \stackrel{cf.b)}{=} \frac{e^2}{2} - \frac{n+1}{2} \cdot I_n - I_n = \frac{e^2}{2} - I_n \left(\frac{n+1}{2} + 1 \right) = \frac{e^2}{2} - I_n \frac{n+3}{2} \end{cases}$ $\Rightarrow \frac{e^2}{2} - I_n \frac{n+3}{2} \leq 0 \Rightarrow I_n \frac{n+3}{2} \geq \frac{e^2}{2} \Rightarrow I_n \geq \frac{e^2}{n+3} (1)$ <p>Analog avem:</p> $\begin{cases} I_{n-1} - I_n \geq 0 \\ I_{n-1} - I_n \stackrel{cf.b)}{=} \frac{e^2}{n} - \frac{2}{n} \cdot I_n - I_n = \frac{e^2}{n} - I_n \left(\frac{2}{n} + 1 \right) = \frac{e^2}{n} - I_n \frac{n+2}{n} \end{cases}$ $\Rightarrow \frac{e^2}{n} - I_n \frac{n+2}{n} \geq 0 \Rightarrow I_n \frac{n+2}{n} \leq \frac{e^2}{n} \Rightarrow I_n \leq \frac{e^2}{n+2} (2)$ $\Rightarrow \frac{e^2}{n+3} \leq I_n \leq \frac{e^2}{n+2}$	2p 1p